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Positive and Negative Charged Rods Alternating Along a Line: Exact Results

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The Coulomb system consisting of an equal number of positive and negative charged rods confined to a line with the charges alternating in sign along the line is considered. By replacing the line with a lattice, one can calculate the grand partition function and correlations exactly for one value of the coupling constant. The exact solution exhibits features forbidden in the corresponding continuous system, in which each pair of oppositely charged rods also interact via a short-range repulsive potential, and there is no restriction on the ordering of the charges. The sum rule indicating the phase of the system is identified.

KEY WORDS: Kosterlitz–Thouless phase transition; two-component plasma.

1. INTRODUCTION AND SUMMARY

The two-species, log-potential Coulomb gas was first studied because of an analogy with the Kondo problem.[1,2] Specifically, one was led to consider a system consisting of an equal number of positive and negative charged rods confined to a circle and arranged so that the charges alternate in sign around the circle. In addition to the logarithmic potential between the charged rods, a short-range repulsive potential (range τ, say) is needed so that the system does not collapse at low temperatures. The Coulomb gas is then a two-parameter system, characterized by the coupling constant $\Gamma \equiv q^2/k_B T$ (q is the magnitude of the charges) and the quantity $\rho\tau$, which is the ratio of the range of the short-range repulsive potential τ to the interparticle spacing $1/\rho$.

In a mapping valid in the low-density, $\rho\tau \to 0$ limit, the free energy and dipole moment of the Coulomb gas were related to the ground-state energy

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and susceptibility, respectively, of the Kondo problem. Furthermore, the length of the circle containing the charges is inversely related to the temperature in the Kondo problem, so the thermodynamic limit of the Coulomb gas reproduces the zero-temperature properties of the Kondo problem. We will be primarily interested in this case.

As noted above, motivated by this analogy, two studies were made of the Coulomb gas. The first was by Anderson *et al.*[1] They developed a scaling theory whereby $\rho\tau$ is varied infinitesimally and the resulting grand partition function related to that of the same system with the original value of $\rho\tau$, but a different Γ. In fact, the scaling theory leads to a set of coupled differential equations for the free energy, Γ, and $\rho\tau$. These equations have a fixed point at the value of the coupling constant $\Gamma = 2$. This is a transition point (again we stress that such analysis can only be justified in the low-density, $\rho\tau \to 0$ limit.) In the original Kondo problem it corresponds to a transition from antiferromagnetic ($\Gamma < 2$) to ferromagnetic ($\Gamma > 2$) coupling of the impurity spin.

The two phases of the Kondo problem can be characterized by the susceptibility χ. For low temperatures T in the Kondo problem the susceptibility in the ferromagnetic regime behaves as

$$\chi \sim 1/T \tag{1.1}$$

while in the antiferromagnetic regime

$$\chi \sim \text{const} > 0 \tag{1.2}$$

In the second of the aforementioned studies of the Coulomb gas, it was shown by Schotte and Schotte[2] that χ is related to the dipole moment of the Coulomb gas. Specifically,

$$\chi = \frac{1}{L} \left\langle \left(\sum_{i=1}^{N} q_i x_i - \frac{1}{2} L \right)^2 \right\rangle \tag{1.3}$$

where L is the circumference of the circle containing the Coulomb gas. But since L is proportional to $1/T$, we have from (1.1) and (1.2) the results

$$\left\langle \left(\sum q_i x_i - L/2 \right)^2 \right\rangle \sim L \qquad \text{for} \quad \Gamma < 2 \tag{1.4}$$

$$\left\langle \left(\sum q_i x_i - L/2 \right)^2 \right\rangle \sim L^2 \qquad \text{for} \quad \Gamma > 2 \tag{1.5}$$

Thus we are already in possession of some information regarding the

Coulomb gas, but for reasons to be discussed below, it is not straightforward to interpret these results.

It is our purpose to investigate this Coulomb gas more closely. This study will be possible because of an exact solvability property of the Coulomb gas: if we put the system on a lattice, the free energy and n-particle correlation functions can be calculated in closed form at the coupling $\Gamma=1$. (In the original Kondo problem this corresponds to the exactly solvable Thouless limit.) By calculating the two- and three-particle correlation functions away from the ends of the system (even though the domain is a circle, the system is not translation invariant, because of the specified ordering of the charges), two special properties of this system are demonstrated.

Firstly, at $\Gamma=1$ the system is not in a conducting state, which means the charges are not free to respond to and screen an external charge density in the long wavelength limit. This in itself is not very surprising, since the specified ordering of the charges prevents redistribution of the charge density to exactly screen an arbitrary charge density with total charge $\neq N_q$, N an integer.

Secondly, even though the state is not conducting, the two-particle correlations decay as $1/x^4$ (x denoting the interparticle distance), the three-particle correlations decay at least as fast as $1/x^3$, and the dipole moment of the screening cloud about any two fixed charges within the system vanishes. This contrasts with the allowed behavior of a continuous system satisfying the first and second BGY equations (and thus not having any restrictions on the ordering of the charges). For then, precisely because of the above conditions on the decay of the correlations and the validity of the dipole moment sum rule imply that the system must be in a conducting state (Alastuey and Martin[3]).

In the last section we discuss these results with respect to the phase of the system. In particular we identify the dipole moment sum rule as the phase indicator.

2. EXACT SOLUTION AT $\Gamma=1$

To properly define the problem, we need to make an explicit choice of the short-range cutoff. For the exact solvability property we need to choose a lattice for this purpose.

Divide a line of length L into M intervals so that there are sites at the points nL/M, $n=1, 2,..., M$. Introduce an interlacing lattice at the points $(n-1/2)L/M$, $n=1, 2,..., M$. Denote these lattices $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. Allow N ($\leqslant M$) positive charges to occupy $\mathscr{L}_1$ and N negative charges to occupy $\mathscr{L}_2$. Impose periodic boundary conditions so that the pair potential is

$$V(\theta_1, \theta_2) = -q_1 q_2 \log[|e^{2\pi i\theta_1/L} - e^{2\pi i\theta_2/L}|\,(L/2\pi)] \tag{2.1}$$

(this is equivalent to defining the system on a circle of circumference L).

Denote the coordinates of the positive charges by $m_k L/M$ and the coordinates of the negative charges by $(l_k - 1/2)L/M$, m_k, $l_k = 1, 2,..., M$. Further denote

$$w_k = e^{2\pi i m_k/M}, \qquad z_k = e^{2\pi i(l_k - 1/2)/M} \tag{2.2}$$

With this notation and the ordering of the charges

$$\begin{gathered} 1 \leqslant l_1 \leqslant m_1, \qquad m_1 + 1 \leqslant l_2 \leqslant m_2,..., m_{N-1} + 1 \leqslant l_N \leqslant m_N \\ 1 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_N \leqslant M \end{gathered} \tag{2.3}$$

so that they alternate in sign, the Boltzmann factor of the system for general $\Gamma \equiv q^2/k_B T$ is

$$W_\Gamma = (2\pi/L)^{\Gamma N} \left(\prod_{1 \leqslant j \leqslant k \leqslant N} |w_k - w_j|\,|z_k - z_j| \bigg/ \prod_{j=1}^{N} \prod_{k=1}^{N} |z_k - w_j| \right)^\Gamma \tag{2.4}$$

We will now proceed to transform (2.4) into a manageable expression for $\Gamma = 1$. Using the identity

$$|e^{i\theta_j} - e^{i\theta_k}| = i^{-1}\, e^{-(\theta_j + \theta_k)/2}(e^{i\theta_j} - e^{i\theta_k}), \qquad \theta_j \geqslant \theta_k \tag{2.5}$$

and the Cauchy double alternant determinant formula

$$\det[(1 - w_j z_k)^{-1}] = \prod_{1 \leqslant j < k \leqslant N} (w_k - w_j)(z_k - z_j) \bigg/ \prod_{j=1}^{N} \prod_{k=1}^{N} (1 - z_j w_k) \tag{2.6}$$

we have

$$W_1 = (-2\pi i/L)^N \left(\prod_{k=1}^{N} e^{\pi i(m_k - l_k + 1/2)/M} \right) \mathrm{Det}[(1 - e^{2\pi i(m_j - l_k + 1/2)/M})^{-1}] \tag{2.7}$$

If we introduce a parameter μ, $|\mu| \leqslant 1$, as a factor of the exponent in each term of the determinant, they can each be Taylor-expanded, and after familiar manipulation[4,5] we obtain

$$\begin{aligned} W_1 = (-2\pi i/L)^N \lim_{\mu \to 1^-} &\sum_{0 \leqslant \alpha_1,...,\alpha_N}^{\infty} \left(\prod_{j=1}^{N} \mu^{\alpha_j} \right) \\ &\times \mathrm{Det}[e^{2\pi i m_j(\alpha_j + 1/2)/M}\, e^{-2\pi i(l_j - 1/2)(\alpha_k + 1/2)/M}] \end{aligned} \tag{2.8}$$

If we write

$$\alpha_j = \gamma_j + k_j M, \qquad 0 \leqslant \gamma_j \leqslant M-1, \qquad k_j = 0, 1, 2,... \tag{2.9}$$

it is straightforward to take the limit $\mu \to 1^-$ and we obtain as our working identity

$$W_1 = (\pi i/L)^N \sum_{0 \leqslant \gamma_1,...,\gamma_N}^{M-1} \operatorname{Det}[e^{2\pi i m_j(\gamma_j + 1/2)/M} e^{-2\pi i (l_j - 1/2)(\gamma_k + 1/2)/M}] \tag{2.10}$$

2.1. Grand Partition Function

For this system the partition function Z_N is given by

$$Z_N = \sum_X W_1 \tag{2.11}$$

where X denotes the restricted range (2.3), and the grand partition function is given by

$$\Xi = \sum_{k=0}^{M} \zeta^{2k} Z_N \tag{2.12}$$

where ζ denotes the activity.

To calculate Z_N, we first sum over the l_k in order, $k = 1, 2,..., N$. From the structure of the determinant (2.10), the sums can be performed row-by-row. After summing over l_1 from 1 to m_1 in the first row, we proceed to sum over l_2 from $m_1 + 1$ to m_2 in the second row. But

$$\sum_{l_2 = m_1+1}^{m_2} = \sum_{l_2=1}^{m_2} - \sum_{l_2=1}^{m_1} \tag{2.13}$$

and since the summand is the same as in the first row, by adding an appropriate multiple of the first row to the second row we are left with

$$\begin{aligned} &e^{2\pi i m_2(\gamma_2 + 1/2)} \sum_{l=1}^{m_2} e^{-2\pi i(l-1/2)(\gamma_k+1/2)/M} \\ &\quad = e^{2\pi i m_2(\gamma_2+1/2)}(1 - e^{-2\pi i m_2(\gamma_k + 1/2)/M})[2i \sin \pi(\gamma_k + 1/2)/M]^{-1} \end{aligned} \tag{2.14}$$

for the second row. Proceeding similarly in each of the remaining sums $l_3,..., l_N$, we obtain

$$\begin{aligned} Z_N = \left(-\frac{\pi}{2L}\right)^N \sum_{1 \leqslant m_1 \leqslant \cdots \leqslant m_N \leqslant N} &\left\{ \sum_{0 \leqslant \gamma_1,...,\gamma_N}^{M-1} \left(\prod_{l=1}^{N} \frac{1}{\sin \pi(\gamma_l + 1/2)/M} \right) \right. \\ &\left. \times \operatorname{Det}[e^{2\pi i m_j(\gamma_j + 1/2)/M}(1 - e^{-2\pi i m_j(\gamma_k+1/2)/M})] \right\} \end{aligned} \tag{2.15}$$

The expression within the brackets $\{\ \}$ is a symmetric function of the m's. To see this, suppose we interchange m_l and $m_{l'}$. But interchanging γ_l and $\gamma_{l'}$ clearly leaves the value of this expression unchanged, and then interchanging both the lth and l'th rows and the lth and l'th columns also leaves the value of the expression unchanged and returns it to its original form. Thus, we can drop the ordering constraint on the sum over the m's provided we divide by $N!$

The m_j can now be summed over from 1 up to M row-by-row. Noting

$$\sum_{m=1}^{M} e^{2\pi i m(\gamma_j+1/2)/M} - e^{2\pi i m(\gamma_j-\gamma_k)/M}$$
$$= -2/(1-e^{-2\pi i(\gamma_j+1/2)/M}) - M\delta_{\gamma_j-\gamma_k,0} \tag{2.16}$$

where the δ denotes the Kronecker delta, we obtain

$$Z_N = \left(\frac{\pi}{2L}\right)^N \frac{1}{N!} \sum_{0\leqslant\gamma_1,\ldots,\gamma_N}^{M-1} \left(\prod_{l=1}^{N} \frac{1}{\sin\pi(\gamma_l+1/2)/M}\right)$$
$$\times \operatorname{Det}[2(1-e^{-2\pi i(\gamma_j+1/2)/M})^{-1} + M\delta_{\gamma_j-\gamma_k,0}] \tag{2.17}$$

If $\gamma_j=\gamma_k$, $j\neq k$, then in the above determinant two rows are the same, so we can choose $\gamma_j\neq\gamma_k$. Therefore, since the expression is symmetric in the γ's,

$$Z_N = \left(\frac{\pi}{2L}\right)^N \sum_{0\leqslant\gamma_1<\cdots<\gamma_N\leqslant M-1} \left(\prod_{l=1}^{N} \frac{1}{\sin\pi(\gamma_l+1/2)/M}\right)$$
$$\times \operatorname{Det}[M\delta_{j-k,0} + 2(1-e^{-2\pi i(\gamma_j+1/2)/M})^{-1}] \tag{2.18}$$

To evaluate this determinant, we first take out the factor

$$M/\lambda_j \equiv 2/(1-e^{-2\pi i(\gamma_j+1/2)/M})$$

from the jth row and then use the readily derived identity

$$\det[a_{ij}] = \prod_{j=1}^{N} \lambda_j + \sum_{k=1}^{N} \prod_{\substack{j=1\\ j\neq k}}^{N} \lambda_j \tag{2.19}$$

where $a_{ij}=1+\lambda_j$ for $i=j$ and 1 otherwise. We thus have evaluated Z_N, with the result

$$Z_N = (\pi M/2L)^N \sum_{0 \leqslant \gamma_1 < \cdots < \gamma_N \leqslant M-1} \left(\prod_{l=1}^{N} \frac{1}{\sin \pi(\gamma_l + 1/2)/M} \right)$$
$$\times \left[1 + \frac{2}{M} \sum_{j=1}^{N} (1 - e^{-2\pi i(\gamma_j + 1/2)/M})^{-1} \right]$$
$$= \text{coefficient of } \zeta^{2N} \text{ in the expansion of the function}$$
$$\left[\prod_{l=0}^{M-1} \left(1 + \frac{\zeta^2 \pi M}{2L \sin \pi(l + 1/2)/M} \right) \right]$$
$$\times \left(1 + \frac{\pi\zeta^2}{2L} \sum_{j=0}^{M-1} \frac{1}{\sin \pi(j + 1/2)/M + \pi\zeta M/2L} \right) \tag{2.20}$$

From (2.12) we see that this expression is in fact the grand partition function Ξ. Hence the pressure P is given by

$$\beta P \equiv \lim_{L \to \infty} \frac{1}{L} \log \Xi$$
$$= \frac{1}{\tau} \int_0^1 dt \log \left(1 + \frac{\pi\zeta^2}{2\tau \sin \pi t} \right) \tag{2.21}$$

where

$$\tau = L/M \tag{2.22}$$

is the lattice spacing.

2.2. Correlaltion Functions

Within the domain from zero to L the charges are ordered so that a negative charge is closest to the boundary at zero and a positive charge closest to the boundary at L. Thus, even with periodic boundary conditions the system is not translation invariant, nor is it invariant under charge negation. For example, the one-particle correlation is dependent on the position within the system and is different for each species of charge. However, we expect this effect to be confined to the neighborhoods of the boundaries. If the correlations are defined so that the test charges are located near the center of the system and then the thermodynamic limit is taken, the correlations will be translation invariant and have charge negation symmetry. We consider this case only in our exact results.

We will be interested in calculating both the two- and three-particle correlation functions. Let us illustrate the procedure by giving some details of the calculation for the two-particle correlation between unlike charges. With $l_a - 1/2$ denoting the coordinate of a fixed negative charge and m_a the coordinate of a fixed positive charge, this correlation is defined as

$$\rho_{-,+}(l_a - 1/2, m_a) = \Xi^{-1} \sum_{n=0}^{M-1} \zeta^{2n} Z(l_a, m_a) \tag{2.23}$$

where

$$Z(l_a, m_a) = \frac{\delta^2}{\delta a(l_a)\, \delta b(m_a)} \sum_X \left\{ \prod_{k=1}^{N} [1 + a(l_k)][1 + b(m_k)] \right\} W_1 \Bigg|_{a=b=0} \tag{2.24}$$

Here X denotes the summation ranges (2.3), and $\delta/\delta a$ and $\delta/\delta b$ denote functional differentiation.

Using the expression for W_1 in (2.10), we first sum over the l's row-by-row, canceling terms as noted in the paragraph around (2.13), and then sum over the m's (again the summand is symmetric in the m's, so the ordering constraint can be dropped provided we divide by $N!$). This gives

$$Z(l_a, m_a) = \frac{1}{N!} \left(\frac{i\pi}{L}\right)^N \sum_{0 \leqslant \gamma_1, \ldots, \gamma_N}^{M-1} \operatorname{Det}\left[\sum_{m=1}^{M} \sum_{l=1}^{m} [1 + a(l)][1 + b(l)] \times e^{2\pi i m_j(\gamma_j + 1/2)/M}\, e^{-2\pi i(l_j - 1/2)(\gamma_k + 1/2)} \right] \tag{2.25}$$

The functional differentiation can now be performd by differentiating the determinant row-by-row. Each determinant thus obtained is in fact the same, as can be seen by interchanging appropriate rows, columns, and summation labels. Hence

$$Z(l_a, m_a) = \frac{1}{(N-2)!} \left(\frac{\pi}{2L}\right)^N \sum_{0 \leqslant \gamma_1, \ldots, \gamma_N}^{M-1} \left(\prod_{l=1}^{N} \frac{1}{\sin \pi(\gamma_l + 1/2)/M} \right) \times \left[\prod_{j=1}^{N-2} 2(1 - e^{-2\pi i(\gamma_j + 1/2)/M})^{-1} \right] \det[b_{jk}] \tag{2.26}$$

where

$$b_{jk} = \begin{cases} 1 + \lambda_j, & j = k, \quad k \neq N-1, N-2 \\ 1, & j \neq k, \quad k \neq N-1, N-2 \\ \psi_k, & j = N-1 \\ \phi_k, & j = N \end{cases} \tag{2.27}$$

and

$$\psi_k = -e^{-2\pi i(l_a-1/2)(\gamma_k+1/2)/M}\left(e^{2\pi i(l_a-1/2)(\gamma_{N-1}+1/2)/M}+e^{\pi i(\gamma_{N-1}+1/2)/M}\right)$$
$$\times\frac{\sin\pi(\gamma_k+1/2)/M}{\sin\pi(\gamma_{N-1}+1/2)/M} \tag{2.28}$$

$$\phi_k = e^{2\pi i m_a(\gamma_N+1/2)/M}\left(1-e^{-2\pi i m_a(\gamma_k+1/2)/M}\right) \tag{2.29}$$

The λ_j are defined in the paragraph between (2.18) and (2.19). By elementary row operations we readily find

$$\mathrm{Det}[b_{jk}] = \left(\prod_{l=1}^{N-2}\lambda_l\right)\left[f_0(\gamma_{N-1},\gamma_N)+\sum_{v=1}^{3}f_v(\gamma_{N-1},\gamma_N)\sum_{k=1}^{N-2}g_v(\gamma_k)\right] \tag{2.30}$$

where

$$\begin{gathered} f_0=f_1=(\phi_N\psi_{N-1}-\psi_N\phi_{N-1}) \\ f_2=\phi_{N-1}-\phi_N, \qquad f_3=\psi_N-\psi_{N-1} \\ g_1=1/\lambda_k, \qquad g_2=\psi_k/\lambda_k, \qquad g_3=\phi_k/\lambda_k \end{gathered} \tag{2.31}$$

Inserting the functional form (2.30) in (2.26) and using the definition (2.23), we have

$$\begin{aligned} &\rho_{-,+}\left(l_a-\frac{1}{2},m_a\right) \\ &\quad=\frac{1}{\Xi}\left(\frac{\pi\zeta^2}{2L}\right)^2\prod_{l=0}^{M-1}\left(1+\frac{\pi\zeta^2}{2\tau\sin\pi(l+1/2)/M}\right) \\ &\quad\times\sum_{\gamma_{N-1}=0}^{M-1}\sum_{\gamma_N=0}^{M-1}\left(f_0+\frac{\pi\zeta^2}{2L}\sum_{v=1}^{3}f_v\sum_{\substack{k=0\\k\neq\gamma_{N-1},\gamma_N}}^{M-1}\frac{g_v(k)}{\sin\pi(k+1/2)/M+\pi\zeta^2M/2L}\right) \\ &\quad\times\left(\sin\left[\frac{\pi(\gamma_{N-1}+1/2)}{M}\right]+\frac{\pi\zeta^2}{2\tau}\right)^{-1}\left(\sin\left[\frac{\pi(\gamma_N+1/2)}{M}\right]+\frac{\pi\zeta^2}{2\tau}\right)^{-1} \end{aligned} \tag{2.32}$$

It now remains to insert the explicit form of the f's, g's, and Ξ, and note that in the thermodynamic limit the sums become integrals. Recalling that the test charges at m_a and $l_a-1/2$ are located near the center of the system, we see that only functions of m_a-l_a contribute. The final result is

$$\begin{aligned} \rho_{-,+}\left(l_a-\frac{1}{2},m_a\right) &= \rho^2+\frac{S_0(l_a-m_a-1/2)\,S_2(l_a-m_a-1/2)}{\pi\zeta^2/2\tau} \\ &\quad-\left[S_2\left(l_a-m_a-\frac{1}{2}\right)\right]^2 \end{aligned} \tag{2.33}$$

where

$$S_0(x)=\frac{\pi\zeta^2}{2\tau}\int_0^1 dt\sin 2\pi xt \tag{2.34}$$

$$S_2(x)=\frac{\pi\zeta^2}{2\tau}\int_0^1 \frac{dt\sin 2\pi xt}{\sin\pi t+\pi\zeta^2/2\tau} \tag{2.35}$$

Although this result was derived with the assumption $l_a > m_a$, it also holds for $l_a \leqslant m_a$ (and thus has charge negation symmetry).

By proceeding similarly we obtain the further evaluations

$$\rho_{+,+}(m_a, m_b)=\rho^2-[C_2(m_a-m_b)]^2 \tag{2.36}$$

$$\begin{aligned}\rho_{+,+,+}(m_a, m_b, m_c)=\rho^3+\rho[&\rho_{+,+}(m_a, m_b)+\rho_{+,+}(m_b, m_c)\\ &+\rho_{+,+}(m_c, m_a)]\\ &+2C_2(m_a-m_b)\,C_2(m_b-m_c)\,C_2(m_c-m_a)\end{aligned} \tag{2.37}$$

$$\begin{aligned}&\rho_{-,+,+}(l_a-1/2, m_a, m_b)\\ &\quad=\rho^3+\rho[\rho_{-,+}(l_a-1/2, m_a)+\rho_{-,+}(l_a-1/2, m_b)+\rho_{+,+}(m_a, m_b)]\\ &\qquad+C_2(m_a-m_b)\left[2S_2(l_a-m_a-1/2)\,S_2(l_a-m_b-1/2)\right.\\ &\qquad-\frac{S_2(l_a-m_b-1/2)\,S_0(l_a-m_a-1/2)}{\pi\zeta^2/2\tau}\\ &\qquad\left.-\frac{S_0(l_a-m_b-1/2)\,S_2(l_a-m_a-1/2)}{\pi\zeta^2/2\tau}\right]\end{aligned} \tag{2.38}$$

where

$$C_2(x)=\frac{\pi\zeta^2}{2\tau}\int_0^1 \frac{dt\cos 2\pi xt}{\sin\pi t+\pi\zeta^2/2\tau} \tag{2.39}$$

All other correlations can be deduced from these results by charge negation symmetry. (As noted above, the original system does not have charge negation symmetry. However, by explicit calculation, the correlations in the interior of the system do have this property, as expected.)

3. SUM RULES

3.1. Perfect Screening Sum Rule

A good check on the accuracy of our working is to verify the perfect screening sum rule,[6] since it is thought to be equivalent to the existence of the thermodynamic limit. This says that the amount of charge contained within the screening cloud of any charge (or group of charges) in the system is equal and opposite to that of the charge (or group of charges). Thus

$$-\sum_{l=-\infty}^{\infty} \rho_{-,+}(l-1/2, m_a) + \sum_{m_b=-\infty}^{\infty} \rho_{+,+}(m_b, m_a) = -\rho \tag{3.1}$$

$$\begin{aligned}-\sum_{l=-\infty}^{\infty} \rho^T_{-,+,+}(l-1/2, m_a, m_b) + \sum_{m=-\infty}^{\infty} \rho^T_{+,+,+}(m, m_a, m_b)\\ = -2\rho^T_{+,+}(m_a, m_b)\end{aligned} \tag{3.2}$$

where the T denotes the appropriate (two- or three-particle) truncated distribution function. From the exact results (2.33) and (2.36)–(2.38), the screening sum rules (3.1) and (3.2) are readily verified.

3.2. Asymptotic Behavior of the Correlations

In the continuous log-potential Coulomb system confined to a line, the state of the system (conducting or insulating) can be determined by the asymptotic behavior of the charge–charge correlation function.[7,8] This sum rule is the analogue of the Stillinger–Lovett sum rule for log-potential systems in a two-dimensional domain. A similar sum rule holds in the present case.

Recall that a conducting state is characterized by its ability to screen an external charge density $\delta\rho_{\text{ext}} = \lambda e^{ikx}$ in the long-wavelength, $k \to 0$ limit. In the lattice case this means that the change in charge density on neighboring lattice sites (one available to positive charges, the other negative charges) will exactly compensate the amount of charge within a region of length τ (the lattice spacing) in the external charge density.

With $\langle \cdot \rangle_\lambda$ denoting the canonical average in the presence of the external charge density, $C_-(l-1/2)$ the microscopic charge density at point $l-1/2$ of the negative charges, and $C_+(m)$ the microscopic charge density at point m of the positive charges, our characterization of the conducting state reads

$$\begin{aligned}&[\langle C_-(l-1/2)\rangle_\lambda + \langle C_+(l)\rangle_\lambda] - [\langle C_-(l-1/2)\rangle_{\lambda=0} + \langle C_+(l)\rangle_{\lambda=0}]\\ &\quad\sim -\lambda\tau e^{ik\tau l} \qquad \text{as} \quad k \to 0\end{aligned} \tag{3.3}$$

We now use a linear response relation, which equates the left-hand side of (3.3) to

$$\begin{aligned}\lambda\beta\{&\langle [C_-(l-1/2) + C_+(l)] H_1\rangle_{\lambda=0}\\ &- \langle [C_-(l-1/2) + C_+(l)]\rangle_{\lambda=0} \langle H_1\rangle_{\lambda=0}\}\end{aligned} \tag{3.4}$$

Here H_1, the additional Hamiltonian due to the external charge density, is given by

$$\lambda H_1 = \frac{\lambda\pi}{|k|} \sum_{m=-\infty}^{\infty} [e^{i\tau m} C_+(m) + e^{i\tau(m-1/2)} C_-(m-1/2)] \tag{3.5}$$

After writing the canonical averages in (3.4) as truncated two-particle distributions and then equating to the right-hand side of (3.3), we obtain the $k \to 0$, leading order singular behavior sum rule

$$2 \sum_{m=-\infty}^{\infty} e^{ikm\tau} [\rho\delta_{m,0} + \rho^T_{+,+}(|m|) - \rho^T_{+,-}(|m-1/2|)] \sim \frac{\tau|k|}{\pi\Gamma} \tag{3.6}$$

From the theory of Fourier series[9] this is equivalent to saying that for large $|m|$

$$2[\rho^T_{+,+}(|m|) - \rho^T_{+,-}(|m-1/2|)] \sim \frac{-1}{\pi^2\Gamma|m|^2} \tag{3.7}$$

From the exact expressions (2.33) and (2.36)–(2.38) we readily deduce the large-distance decay of the correlations

$$\rho^T_{+,+}(m_a, m_b) \sim -\tau^2 \left(\frac{1}{\pi\zeta(m_a - m_b)}\right)^4 \quad \text{as} \quad |m_a - m_b| \to \infty \tag{3.8}$$

$$\rho^T_{-,+}\left(l_a - \frac{1}{2}, m_a\right) \sim \tau^2 \left(\frac{1}{\pi\zeta(l_a - m_a - 1/2)}\right)^4 \quad \text{as} \quad |l_a - m_a| \to \infty \tag{3.9}$$

$$\rho^T_{+,+,+}(m_a, m_b, m_c) \sim 2\tau^2 C_2(m_a - m_b) \left(\frac{1}{\pi\zeta(m_c - m_b)}\right)^2 \left(\frac{1}{\pi\zeta(m_c - m_a)}\right)^2$$
$$\text{for} \quad m_a, m_b \text{ fixed and } |m_c| \to \infty \tag{3.10}$$

$$\rho^T_{-,+,+}\left(l_a - \frac{1}{2}, m_a, m_b\right)$$
$$\sim -\left(\frac{\tau}{\zeta^2}\right)^2 C_2(m_a - m_b) \left(\frac{1}{[\pi(l_a - m_b - 1/2)]^3\, \pi(l_a - m_a - 1/2)} + \frac{1}{[\pi(l_a - m_a - 1/2)]^3\, \pi(l_a - m_b - 1/2)}\right)$$
$$\text{for} \quad m_a, m_b \text{ fixed and } |l_a| \to \infty \tag{3.11}$$

$$\rho^T_{-,+,+}\left(l_a - \frac{1}{2}, m_a, m_b\right)$$
$$\sim \tau\left(\frac{1}{\pi\zeta(m_a - m_b)}\right)^2\left(\frac{S_2(l_a - m_a - 1/2)}{\pi(l_a - m_b - 1/2)} - \frac{S_0(l_a - m_a - 1/2)}{\pi(l_a - m_b - 1/2)}\right)$$
$$\text{for} \quad l_a, m_a \text{ fixed and } m_b \to \infty \tag{3.12}$$

Hence from (3.8) and (3.9)

$$[\rho^T_{+,+}(|m|) - \rho^T_{+,-}(|m - 1/2|)] \sim -2\tau^2(1/\pi\zeta m)^4 \tag{3.13}$$

which contradicts the sum rule (3.7), so we thus conclude that the system is in an insulating state.

3.3. Dipole Moment Sum Rule

Suppose we fix two charges in the system and compare the charge density after this even to before (that is, we consider the screening cloud of the fixed charges). The dipole moment sum rule says that the dipole moment of this screening cloud will vanish.

Unlike the perfect screening sum rule, this sum rule is not thought to be generally true. Rather, by an analysis of the BGY equations, it apears for continuous systems that this sum rule breaks down if the system is not in a conducting state.[3,10] (By saying this, we are of course assuming that the truncated three-particle distribution functions decay fast enough for the dipole moment to exist.)

From (3.10)–(3.12) the three-particle correlations here decay at least as fast as $O(1/x^3)$, so the dipole moment sum rule is well defined. It states

$$\sum_{m=-\infty}^{\infty} m\rho^T_{+,+,+}(m, m_a, m_b) - \sum_{l=-\infty}^{\infty} (l - 1/2)\,\rho^T_{-,+,+}(l - 1/2, m_a, m_b)$$
$$= -(m_a + m_b)\,\rho^T_{+,+}(m_a, m_b) \tag{3.14}$$

$$\sum_{m'=-\infty}^{\infty} m'\rho^T_{+,-,+}(m', l_a - 1/2, m_a)$$
$$- \sum_{l'=-\infty}^{\infty} (l' - 1/2)\,\rho^T_{-,-,+}(l' - 1/2, l_a - 1/2, m_a)$$
$$= [(l_a - 1/2) - m_a]\,\rho^T_{-,+}(l_a - 1/2, m_a) \tag{3.15}$$

Using the exact expressions (2.33) and (2.36)–(2.38), we find that these sum rules are obeyed. To illustrate our method of calculation, we give the derivation of (3.15) in the Appendix.

4. DISCUSSION

Let us first address ourselves to the question of the phase of the system. We noted in the introduction that it seems clear that the system can never be in a conducting phase in the sense that the sum rule (3.13) will not be satisfied. Nevertheless the scaling theory of Anderson et al.[1] indicates a Kosterlitz–Thouless type phase transition at $\Gamma = 2$. Furthermore, photographs taken in Schotte and Schotte's Monte-Carlo simulation show in visual terms that, for temperatures $\Gamma > 2$, the system consists entirely of dipoles, while the region $2 > \Gamma > 1$ has a domain structure, in which groups of dipoles are separated by a single positive or negative charge (no simulation was performed for $\Gamma < 1$). What sum rule characterizes the particular phase?

To answer this question, we appeal to the analogy with the two-dimensional two-component plasma. Then it is known that the conducting phase can be characterized either by the Stillinger–Lovett or dipole moment sum rule.[10],[3] This is intuitively reasonable since one would think that, with no restriction on the ordering of the charges, both these sum rules would hold if and only if there is a macroscopic number of free charges. We have argued that the analog of the Stillinger–Lovett sum rule (3.13) cannot be satisfied in the present system. Let us consider the dipole moment sum rule. With the test charges being chosen from those in the system, it would seem that here the restricted ordering of the charges plays no essential role. Thus we would expect that the dipole moment sum rule identifies the phase of the system, the rule being valid if and only if there is a macroscopic number of free charges in the system, that is, for $\Gamma < 2$. Our exact result at $\Gamma = 1$ is in agreement with these remarks.

Now consider the results on the dipole moment (1.4) and (1.5). We would like to use these results to deduce the large distance decay of the two-particle correlations. Unfortunately, this is not possible since the canonical average relates to the finite system in which the system does not possess charge negation symmetry, and thus has a non-zero dipole moment. Hence, the canonical averages cannot be related to the second moment of the "bulk" (i.e. translation and charge negation invariant) two-particle correlation. However, these results do make one suspicious of a change in the rate of decay of the bulk two-particle correlation at the $\Gamma = 2$ transition.

Finally, let us speculate on a further feature of our exact solution isotherm $\Gamma = 1$. We have already noted that Schotte and Schotte's computer simulation shows a domain structure for $2 > \Gamma > 1$. It may be that for $\Gamma < 1$ all dipoles in the system have ionized, so that $\Gamma = 1$ is the temperature at which dipoles can first form in the system. If this is true, then the situation would be analogous to the two-dimensional two-component plasma. There, in the low density limit, computer simulation[11] shows that, for $4 > \Gamma > 2$, there is a mixture of dipoles and ions, while for $\Gamma < 2$ the system consists entirely of free charges. Indeed the $\Gamma = 2$ boundary between the two regions is again a solvable isotherm.[12]

APPENDIX

We want to derive the dipole moment sum rule (3.15). First, from the exact expression (2.38) one can check the symmetry relations

$$\rho^T_{+,-,+}(m', l_a - 1/2, m_a) = \rho^T_{+,-,+}(l_a, m_a + 1/2, m_a - m' + l_a) \tag{A1}$$

$$\rho^T_{-,-,+}(l' - 1/2, l_a - 1/2, m_a) = \rho^T_{+,+,-}(l', l_a, m_a + 1/2) \tag{A2}$$

Using these relationships, one finds that the left-hand side of (3.15) becomes

$$\begin{aligned}(l_a + m_a + 1/2) &\sum_{m'=-\infty}^{\infty} \rho^T_{+,+,-}(m', l_a, m_a + 1/2) \\ &- 2 \sum_{m'=-\infty}^{\infty} m' \rho^T_{+,+,-}(m', l_a, m_a + 1/2)\end{aligned} \tag{A3}$$

Since

$$\sum_{m=-\infty}^{\infty} e^{2\pi i m(t-s)} = \delta(t-s), \qquad |t-s| < 1 \tag{A4}$$

we obtain, after using the expression (2.38) and interchanging the order of summation and integration, the result

$$\sum_{m'=-\infty}^{\infty} \rho^T_{+,+,-}(m', l_a, m_a + 1/2) = I - [S_2(m_a + 1/2 - l_a)]^2 \tag{A5}$$

where

$$I = \left[2S_2\left(m_a + \frac{1}{2} - l_a\right) - S_0\left(m_a + \frac{1}{2} - l_a\right)\right] \frac{\pi\zeta^2}{2\tau}$$

$$\times \int_0^1 \frac{dt \sin 2\pi(m_a + 1/2 - l_a)t}{(\sin \pi t + \pi\zeta^2/2\tau)^2} \tag{A6}$$

To evaluate the remaining term in (A3), we require the formulas

$$\frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} m e^{2\pi i m(t-s)} = \delta'(t-s), \qquad |t-s| < 1 \tag{A7}$$

$$\int_0^1 ds \int_0^1 dt\, f(t)\, g(s)\, \delta'(t-s)$$

$$= \frac{1}{2} [f(1)\, g(1) - f(0)\, g(0)] - \int_0^1 ds\, g(s)\, f'(s) \tag{A8}$$

Then, after using the expression (2.38) and interchanging the order of summation and integration, we have the result

$$\sum_{m'=-\infty}^{\infty} m' \rho^T_{+,+,-}(m', l_a, m_a + \tfrac{1}{2})$$

$$= \tfrac{1}{2}(m_a + l_a + \tfrac{1}{2}) I - (m_a + \tfrac{1}{2})[S_2(m_a + \tfrac{1}{2} - l_a)]^2$$

$$+ \tfrac{1}{2}(m_a + \tfrac{1}{2} - l_a)\, S_2(m_a + \tfrac{1}{2} - l_a)\, S_0(m_a + \tfrac{1}{2} - l_a) \tag{A9}$$

Substituting (A5) and (A9) in (A3) and recalling the expression (2.33) for $\rho^T_{-,+}(l_a - 1/2, m_a)$, we obtain the sum rule (3.15).

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